TAYLORŮV POLYNOM FUNKCE JEDNÉ PROMĚNNÉ

Napište Taylorův polynom stupně funkce f stupně n v bodě x_0:

1. $f(x) = \ln(1-x)$, $x_0 = 0$, $n = 4$ $[T_4(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}]$

2. $f(x) = \sqrt[5]{5x+1}$, $x_0 = 0$, $n = 4$ $[T_4(x) = 1 + x - 2x^2 + 6x^3 - 21x^4]$

3. $f(x) = \operatorname{tg} x$, $x_0 = 0$, $n = 3$ $[T_3(x) = x + \frac{x^3}{3}]$

4. $f(x) = \operatorname{tg} x$, $x_0 = \frac{\pi}{4}$, $n = 3$ $[T_3(x) = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3]$

5. $f(x) = \sin(x)$, $x_0 = 0$, $n = 6$ $[T_6(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5]$

6. $f(x) = \ln(\frac{1-x}{1+x})$, $x_0 = 0$, $n = 6$ $[T_6(x) = -2x - \frac{2}{3}x^3 - \frac{2}{5}x^5]$

7. $f(x) = x^2\cos(x)$, $x_0 = 0$, $n = 6$ $[T_6(x) = x^2 - \frac{1}{2}x^4 + \frac{1}{24}x^6]$

8. $f(x) = \operatorname{tg}(x)$, $x_0 = 0$, $n = 5$ $[T_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5]$

9. $f(x) = \sqrt{2x+1}$, $x_0 = 0$, $n = 5$ $[T_5(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4 + \frac{7}{8}x^5]$

10. $f(x) = x\mathrm{arcsin}(x)$, $x_0 = 0$, $n = 6$ $[T_6(x) = x^2 + \frac{1}{6}x^4 + \frac{3}{40}x^6]$

11. $f(x) = e^x$, $x_0 = 0$, $n = 5$ $[T_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5]$

12. $f(x) = \arctan(x)$, $x_0 = 0$, $n = 4$ $[T_4(x) = x - \frac{1}{3}x^3]$

13. $f(x) = \ln(x)$, $x_0 = 1$, $n = 6$
$[T_6(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6]$

14. $f(x) = \ln(\sin(x))$, $x_0 = \frac{1}{2}\pi$, $n = 6$
$[T_6(x) = -\frac{1}{2}(x - \frac{1}{2}\pi)^2 - \frac{1}{12}(x - \frac{1}{2}\pi)^4 - \frac{1}{45}(x - \frac{1}{2}\pi)^6]$

15. $f(x) = x^3\ln(x)$, $x_0 = 1$, $n = 5$
$[T_5(x) = x - 1 + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5]$

16. $f(x) = e^{(2x-x^2)}$, $x_0 = 0$, $n = 5$ $[T_5(x) = 1 + 2x + x^2 - \frac{2}{3}x^3 - \frac{5}{6}x^4 - \frac{1}{15}x^5]$

17. $f(x) = (1+x)\ln(\frac{x}{2})$, $x_0 = 2$, $n = 5$
$[T_5(x) = \frac{3}{2}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{192}(x-2)^4 + \frac{1}{320}(x-2)^5]$

18. $f(x) = \sqrt{x}$, $x_0 = 1$, $n = 4$
$[T_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4]$

19. $f(x) = x\arcsin(x) + \sqrt{1-x^2}$, $x_0 = 0$, $n = 4$ $[T_4(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4]$

20. $f(x) = x^2 e^{(1-x)}$, $x_0 = 1$, $n = 4$
$[T_4(x) = 1 + x - 1 - \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{5}{24}(x-1)^4]$

Taylorův polynom - vzorec

$$T_n(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2 +$$
$$+ \frac{f'''(x_0)}{3!}\cdot(x-x_0)^3 + \frac{f^{IV}(x_0)}{4!}\cdot(x-x_0)^4 + \dots \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Pozn.: $3! = 3\cdot 2\cdot 1$
$4! = 4\cdot 3\cdot 2\cdot 1$
$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$

Řešený příklad

1) $f(x) = \ln(1-x)$ $\quad x_0 = 0 \quad n = 4$
$\Downarrow$
potřebuji 4 derivace

$$f(x) = \ln(1-x)$$
$$f'(x) = \frac{1}{1-x}\cdot(-1) = \frac{-1}{1-x}$$
$$f''(x) = \frac{0\cdot(1-x) + 1\cdot(-1)}{(1-x)^2} = \frac{-1}{(1-x)^2}$$
$$f'''(x) = \frac{0\cdot(1-x)^2 + 1\cdot 2\cdot(1-x)\cdot(-1)}{(1-x)^4} = \frac{-2(1-x)}{(1-x)^4} = \frac{-2}{(1-x)^3}$$
$$f^{IV}(x) = \frac{0\cdot(1-x)^3 + 2\cdot 3(1-x)^2\cdot(-1)}{(1-x)^6} = \frac{-6}{(1-x)^4}$$

dosadím do všech derivací i původní funkce za $x = x_0$

$f(x_0) = \ln 1 = 0$ $\qquad f'''(x_0) = \frac{-2}{(1-0)^3} = -2$
$f'(x_0) = \frac{-1}{1-0} = -1$ $\qquad f^{IV}(x_0) = \frac{-6}{(1-0)^4} = -6$
$f''(x_0) = \frac{-1}{(1-0)^2} = -1$

Dosadím do vzorce:

$$T_4(x) = 0 + (-1)\cdot(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{-2}{3!}(x-0)^3 + \frac{-6}{4!}(x-0)^4$$
$$T_4(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

2) $x_0 = 0 \quad n = 4$

$f(x) = (5x+1)^{\frac{1}{5}}$ $\qquad f(x_0) = 1$

$f'(x) = \frac{1}{5}(5x+1)^{-\frac{4}{5}} \cdot 5 = (5x+1)^{-\frac{4}{5}}$ $\qquad f'(x_0) = 1$

$f''(x) = -\frac{4}{5}(5x+1)^{-\frac{9}{5}} \cdot 5 = -4(5x+1)^{-\frac{9}{5}}$ $\qquad f''(x_0) = -4$

$f'''(x) = \frac{36}{5}(5x+1)^{-\frac{14}{5}} \cdot 5 = 36(5x+1)^{-\frac{14}{5}}$ $\qquad f'''(x_0) = 36$

$f^{IV}(x) = -\frac{504}{5}(5x+1)^{-\frac{19}{5}} \cdot 5 = -504(5x+1)^{-\frac{19}{5}}$ $\qquad f^{IV}(x_0) = -504$

$T_4(x) = 1 + 1(x-0) - \frac{4}{2!}(x-0)^2 + \frac{36}{3!}(x-0)^3 - \frac{504}{4!}(x-0)^4$

$T_4(x) = 1 + x - 2x^2 + 6x^3 - 21x^4$

7) $f(x) = x^2 \cos x$

$f'(x) = 2x\cos x + x^2(-\sin x)$

$f''(x) = 2\cos x + 2x(-\sin x) + 2x \cdot (-\sin x) + x^2 \cdot (-\cos x)$

$= 2\cos x - 4x\sin x - x^2\cos x$

$f'''(x) = -2\sin x - (4\sin x + 4x\cos x) - (2x\cos x + x^2(-\sin x))$

$= -6\sin x - 6x\cos x + x^2\sin x$

$f^{IV}(x) = -6\cos x - (6\cos x + 6x(-\sin x)) + 2x\sin x + x^2\cos x$

$= -12\cos x + 8x\sin x + x^2\cos x$

$f^{V}(x) = +12\sin x + 8\sin x + 8x\cos x + 2x\cos x + x^2(-\sin x)$

$= 20\sin x + 10x\cos x - x^2\sin x$

$f^{VI}(x) = 20\cos x + 10\cos x + 10x(-\sin x) - (2x\sin x + x^2\cos x)$

$= 30\cos x - 12x\sin x - x^2\cos x$

$f(x_0) = 0 \qquad f''(x_0) = 2 \qquad f^{IV}(x_0) = -12 \qquad f^{VI}(x_0) = 30$

$f'(x_0) = 0 \qquad f'''(x_0) = 0 \qquad f^{V}(x_0) = 0$

$T_6(x) = \frac{2}{2!}(x-0)^2 - \frac{12}{4!}(x-0)^4 + \frac{30}{6!}(x-0)^6$

$T_6(x) = x^2 - \frac{1}{2}x^4 + \frac{1}{24}x^6$