UŽITÍ DERIVACE

1) Napište rovnici tečny dané funkce v daném bodě:
   a. $f(x)=\frac{x^2+4}{x-1}, T[2,?]$
   b. $f(x)=3x^2-2\sqrt{x}, T[4,?]$
   c. $f(x)=\frac{x+3}{x-1}, T[2,?]$

2) Určete přibližné hodnoty výrazů pomocí diferenciálu:
   a. $f(x)=\frac{x^2+4}{x-1}, x=2{,}01$
   b. $4{,}03^2-\sqrt{4{,}03}$
   c. $f(x)=2\ln(x-3),\ f(3{,}99)$
   d. $\frac{0{,}99+1}{0{,}99-2}$

3) Vypočítejte limity pomocí L'Hospitalova pravidla:
   a. $\lim_{x\to 0}\frac{arctg\,4x}{5x}$
   b. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$
   c. $\lim_{x\to 2}\frac{x^3-2x-4}{x^2-x-2}$
   d. $\lim_{x\to\infty}\frac{e^{4x}}{x^2}$
   e. $\lim_{x\to\infty}\frac{\log_2 x}{x}$
   f. $\lim_{x\to 1+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$
   g. $\lim_{x\to 0+}\sqrt{x}\ln x$
   h. $\lim_{x\to\infty}x^2e^{-x^2}$
   i. $\lim_{x\to\frac{\pi}{2}-}(\sin x)^{tgx}$

4) Určete intervaly, na kterých je funkce rostoucí, klesající, konvexní, konkávní. Dále určete lokální extrémy a inflexní body.

   a. $f(x)=x^2(x-6)$
   b. $f(x)=x^4-x^2$
   c. $f(x)=\frac{x}{(x+1)^2}$
   d. $f(x)=\frac{x^2}{x^2-4}$
   e. $f(x)=\frac{x}{\ln x}$
   f. $f(x)=x+\ln x^2$
   g. $f(x)=x^2e^{-x}$
   h. $f(x)=x^2e^{\frac{1}{x}}$
   i. $f(x)=(x^2+x+1)e^x$

užití derivace

Tečna:

- $f(x) = \frac{x^2+4}{x-1}$ $\quad T[2; ?]$

$$y_0 = \frac{2^2+4}{2-1} = 8$$

$$f' = \frac{2x(x-1)-(x^2+4)}{(x-1)^2} = \frac{2x^2-2x-x^2-4}{(x-1)^2} = \frac{x^2-2x-4}{(x-1)^2}$$

VZOREC:
$y = f'(x_0)\cdot(x-x_0)+y_0$

$$f'(2) = \frac{4-4-4}{(2-1)^2} = -4$$

$$t:\ y = -4\cdot(x-2)+8$$

$$\underline{\underline{y = -4x+16}}$$

- $f(x) = 3x^2 - 2\sqrt{x}$ $\quad T[4; ?]$

$$y_0 = 3\cdot 4^2 - 2\sqrt{4} = 48 - 4 = 44$$

$$f' = 6x - 2\cdot\frac{1}{2}x^{-\frac{1}{2}} = 6x - \frac{1}{\sqrt{x}}$$

$$f'(4) = 6\cdot 4 - \frac{1}{\sqrt{4}} = 24 - \frac{1}{2} = \frac{47}{2}$$

$$t:\ y = \frac{47}{2}(x-4)+44$$

$$y = \frac{47}{2}x - 94 + 44$$

$$\underline{\underline{y = \frac{47}{2}x - 50}}$$

- $f(x) = \frac{x+3}{x-1}$ $\quad T[2; ?]$

$$y_0 = \frac{2+3}{2-1} = 5$$

$$f' = \frac{1(x-1)-(x+3)\cdot 1}{(x-1)^2} = \frac{-4}{(x-1)^2}$$

$$f'(x_0) = -\frac{4}{(2-1)^2} = -4$$

$$t:\ y = -4(x-2)+5$$

$$\underline{\underline{y = -4x+13}}$$

diferenciál

- $f(x) = \frac{x^2+4}{x-1}$ $\quad x = 2{,}01 \quad x_0 = 2 \rightarrow y_0 = \frac{2^2+4}{2-1} = 8$

$$f' = \frac{2x(x-1)-(x^2+4)}{(x-1)^2} = \frac{2x^2-2x-x^2-4}{(x-1)^2} = \frac{x^2-2x-4}{(x-1)^2}$$

$$f'(2) = \frac{2^2-2\cdot 2-4}{(2-1)^2} = -4$$

$$y = -4\cdot(x-2)+8$$

$$y = -4\cdot(2{,}01-2)+8 = -4\cdot 0{,}01+8 = \underline{\underline{7{,}96}}$$

- $f(x) = x^2 - \sqrt{x}$ $\quad x = 4{,}03 \quad x_0 = 4 \rightarrow y_0 = 4^2-\sqrt{4} = 14$

$$f' = 2x - \frac{1}{2}x^{-\frac{1}{2}} = 2x - \frac{1}{2\sqrt{x}}$$

$$f'(4) = 2\cdot 4 - \frac{1}{2\sqrt{4}} = 8 - \frac{1}{4} = \frac{31}{4}$$

$$y = \frac{31}{4}(x-4)+14$$

$$y = \frac{31}{4}(4{,}03-4)+14 = \underline{\underline{14{,}2325}}$$

- $f(x) = 2\cdot\ln(x-3)$ $\quad x = 3{,}99 \quad x_0 = 4 \rightarrow y_0 = 2\cdot\ln(4-3) = 0$

$$y' = 2\cdot\frac{1}{x-3}$$

$$f'(4) = 2\cdot\frac{1}{4-3} = 2$$

$$y = 2\cdot(x-4)+0$$

$$y = 2\cdot(3{,}99-4) = \underline{\underline{-0{,}02}}$$

- $f(x) = \frac{x+1}{x-2}$ $\quad x = 0{,}99 \quad x_0 = 1 \rightarrow y_0 = \frac{1+1}{1-2} = -2$

$$f' = \frac{1(x-2)-(x+1)}{(x-2)^2} = \frac{x-2-x-1}{(x-2)^2} = \frac{-3}{(x-2)^2}$$

$$f'(1) = \frac{-3}{(1-2)^2} = -3$$

$$y = -3(x-1)-2$$

$$y = -3(0{,}99-1)-2 = \underline{\underline{-1{,}97}}$$

L'hospitalovo pravidlo

- $\lim\limits_{x\to 0} \dfrac{\operatorname{arctg} 4x}{5x} = \left[\dfrac{0}{0}\right] \overset{LH}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{1+(4x)^2}\cdot 4}{5} = \dfrac{4}{5}$

- $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2} = \left[\dfrac{0}{0}\right] \overset{LH}{=} \lim\limits_{x\to 0} \dfrac{e^x-1}{2x} = \left[\dfrac{0}{0}\right] \overset{LH}{=} \lim\limits_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

- $\lim\limits_{x\to 2} \dfrac{x^3-2x-4}{x^2-x-2} = \left[\dfrac{0}{0}\right] \overset{LH}{=} \lim\limits_{x\to 2} \dfrac{3x^2-2}{2x-1} = \dfrac{10}{3}$

- $\lim\limits_{x\to \infty} \dfrac{e^{4x}}{x^2} = \left[\dfrac{\infty}{\infty}\right] \overset{LH}{=} \lim\limits_{x\to \infty} \dfrac{e^{4x}\cdot 4}{2x} = \left[\dfrac{\infty}{\infty}\right] \overset{LH}{=} \lim\limits_{x\to \infty} \dfrac{e^{4x}\cdot 4\cdot 4}{2} = \infty$

- $\lim\limits_{x\to \infty} \dfrac{\log_2 x}{x} = \left[\dfrac{\infty}{\infty}\right] \overset{LH}{=} \lim\limits_{x\to \infty} \dfrac{\frac{1}{x\ln 2}}{1} = 0$

- $\lim\limits_{x\to 1+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right) = \lim\limits_{x\to 1+} \dfrac{\ln x-(x-1)}{(x-1)\ln x} = \left[\dfrac{0}{0}\right] \overset{LH}{=} \lim\limits_{x\to 1+} \dfrac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}} = \left[\dfrac{0}{0}\right] \overset{LH}{=}$

$\rightarrow -\dfrac{1}{x^2} : \dfrac{x+1}{x^2}$

$\lim\limits_{x\to 1+} \dfrac{-x^{-2}}{\frac{1}{x}+\frac{1\cdot x-(x-1)}{x^2}} = \lim\limits_{x\to 1+} \dfrac{-\frac{1}{x^2}}{\frac{x+x-x+1}{x^2}} = \lim\limits_{x\to 1+} \dfrac{-1\cdot \cancel{x^2}}{\cancel{x^2}\cdot(x+1)} = -\dfrac{1}{2}$

- $\lim\limits_{x\to 0+} \sqrt{x}\cdot \ln x = [0\cdot -\infty] = \lim\limits_{x\to 0+} \dfrac{\ln x}{x^{-\frac{1}{2}}} \overset{LH}{=} \lim\limits_{x\to 0+} \dfrac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} =$

$\dfrac{\sqrt{x^3}}{x} = \dfrac{x^{\frac{3}{2}}}{x} = x^{\frac{3}{2}-1} = x^{\frac{1}{2}}$

$= \lim\limits_{x\to 0+} \dfrac{1}{x} : \left(-\dfrac{1}{2\sqrt{x^3}}\right) = \lim\limits_{x\to 0+} \dfrac{1}{x}\cdot\dfrac{2\sqrt{x^3}}{-1} = \lim\limits_{x\to 0+} -2\sqrt{x} = 0$

- $\lim\limits_{x\to \infty} x^2\cdot e^{-x^2} = [\infty\cdot 0] = \lim\limits_{x\to \infty} \dfrac{x^2}{e^{x^2}} \overset{LH}{=} \lim\limits_{x\to \infty} \dfrac{\cancel{2x}}{e^{x^2}\cdot\cancel{2x}} = \left[\dfrac{1}{\infty}\right] = 0$

- $\lim\limits_{x\to \frac{\pi}{2}-} (\sin x)^{\operatorname{tg} x} = [1^\infty] = \lim\limits_{x\to \frac{\pi}{2}-} e^{\operatorname{tg} x\cdot \ln \sin x} = e^0 = 1$

$\lim\limits_{x\to \frac{\pi}{2}-} \operatorname{tg} x\cdot \ln \sin x = [\infty\cdot 0] = \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\ln \sin x}{(\operatorname{tg} x)^{-1}} \overset{LH}{=} \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\frac{1}{\sin x}\cdot \cos x}{-(\operatorname{tg} x)^{-2}\cdot \frac{1}{\cos^2 x}}$

$= \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{\operatorname{tg}^2 x}\cdot\frac{1}{\cos^2 x}} = \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{\frac{\sin^2 x}{\cancel{\cos^2 x}}}\cdot\frac{1}{\cancel{\cos^2 x}}} = \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}} = \lim\limits_{x\to \frac{\pi}{2}-} \dfrac{\cos x\cdot(-\sin^{\cancel{2}} x)}{\cancel{\sin x}\cdot 1} = 0\cdot(-1) = 0$

monotonie + konvex./konkáv.

- $f(x) = x^2 \cdot (x-6) = x^3 - 6x^2$

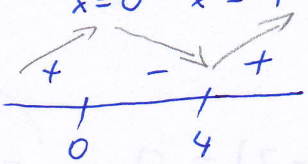
$Df = R$

$f' = 3x^2 - 12x$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$x = 0 \quad x = 4$



rost. $(-\infty; 0\rangle, \langle 4, +\infty)$

kles. $\langle 0, 4\rangle$

lok. min. $[4; -32]$

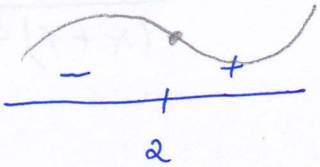
lok. max. $[0; 0]$

$f'' = 6x - 12$

$6x - 12 = 0$

$6x = 12$

$x = 2$



konkávní $(-\infty; 2\rangle$

konvexní $\langle 2, +\infty)$

inflexní bod $[2; -16]$

- $f(x) = x^4 - x^2$

$Df = R$

$f' = 4x^3 - 2x$

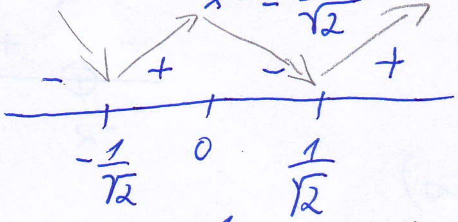
$4x^3 - 2x = 0$

$x(4x^2 - 2) = 0$

$x = 0 \quad 4x^2 = 2$

$x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$



rost. $\langle -\frac{1}{\sqrt{2}}, 0\rangle, \langle \frac{1}{\sqrt{2}}, +\infty)$

kles. $(-\infty; -\frac{1}{\sqrt{2}}\rangle, \langle 0, \frac{1}{\sqrt{2}}\rangle$

lok. min. $[-\frac{1}{\sqrt{2}}; -\frac{1}{4}], [\frac{1}{\sqrt{2}}; -\frac{1}{4}]$

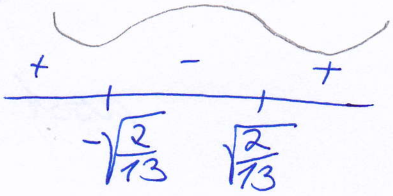
lok. max. $[0; 0]$

$f'' = 13x^2 - 2$

$13x^2 = 2$

$x^2 = \frac{2}{13}$

$x = \pm\sqrt{\frac{2}{13}}$



konvexní $(-\infty; -\sqrt{\frac{2}{13}}\rangle,$ $\langle\sqrt{\frac{2}{13}}, +\infty)$

konkávní $\langle -\sqrt{\frac{2}{13}}, \sqrt{\frac{2}{13}}\rangle$

infl. body: $[-\sqrt{\frac{2}{13}}; \frac{-22}{169}]$

$[\sqrt{\frac{2}{13}}; \frac{-22}{169}]$

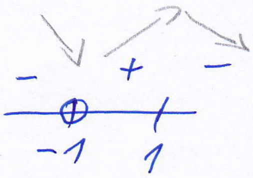
- $f(x) = \frac{x}{(x+1)^2} \quad Df = R \setminus \{-1\}$

$f' = \frac{(x+1)^2 - x\,2(x+1)}{(x+1)^4} = \frac{x^2 + 2x + 1 - 2x^2 - 2x}{(x+1)^4} = \frac{1-x^2}{(x+1)^4}$

$\frac{1-x^2}{(x+1)^4} = 0$

$1 - x^2 = 0$

$x = \pm 1$



rost. $(-1; 1\rangle$

kles. $(-\infty; -1), \langle 1, +\infty)$

lok. max $[1; \frac{1}{4}]$

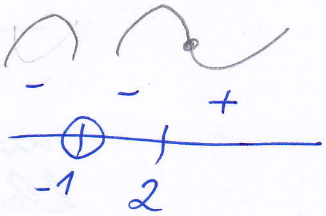
$$f'' = \frac{-2x(x+1)^4 - (1-x^2)(x+1)^3 \cdot 4}{(x+1)^8} = \frac{\cancel{(x+1)^3}\left[-2x(x+1) - 4(1-x^2)\right]}{(x+1)^{\cancel{8}}} =$$

$$= \frac{-2x^2 - 2x - 4 + 4x^2}{(x+1)^5} = \frac{2x^2 - 2x - 4}{(x+1)^5} = \frac{2(x^2 - x - 2)}{(x+1)^5} =$$

$$= \frac{2(x-2)\cancel{(x+1)}}{(x+1)^5} = \frac{2(x-2)}{(x+1)^4}$$

$2(x-2) = 0$

$x = 2$



konvexní $\langle 2; +\infty)$

konkávní $(-\infty; -1), (-1, 2\rangle$

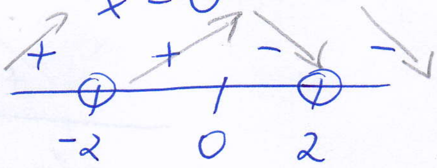
infl. bod $[2; \frac{2}{9}]$

- $f(x) = \frac{x^2}{x^2-4}$ $Df = R \setminus \{-2; 2\}$

$$f' = \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2} = \frac{2x^3 - 8x - 2x^3}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

$-8x = 0$

$x = 0$



rost. $(-\infty; -2)$

$(-2; 0\rangle$

kles. $\langle 0; 2)$

$(2; +\infty)$

lok. max. $[0; 0]$

$$f'' = \frac{-8(x^2-4)^2 + 8x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} = \frac{\cancel{(x^2-4)}\left[-8x^2 + 32 + 32x^2\right]}{(x^2-4)^4} =$$

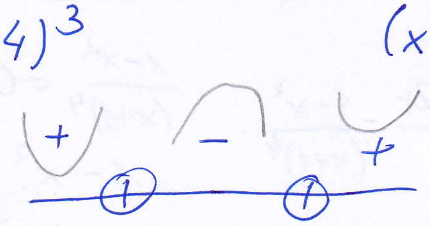
$$= \frac{24x^2 + 32}{(x^2-4)^3} = \frac{8(3x^2+4)}{(x^2-4)^3}$$

$8 \cdot (3x^2 + 4) = 0$

$3x^2 = -4$

$x^2 \neq -\frac{4}{3}$

nelze aby x^2 bylo záporné



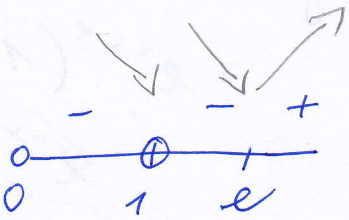
konkávní $(-2, 2)$

konvexní $(-\infty; -2), (2; +\infty)$

- $f(x) = \frac{x}{\ln x}$ $D f = (0; +\infty) - \{1\}$

$\ln x \neq 0$
$x \neq e^0 \neq 1$

$f' = \frac{\ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - 1}{\ln^2 x}$ $\ln x = 1$
$x = e$

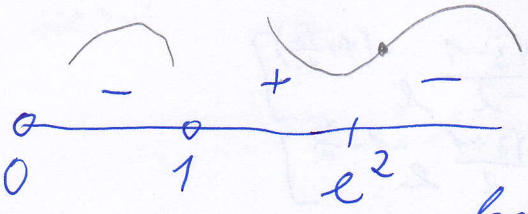


kles. $(0,1), (1, e\rangle$
rost. $\langle e, +\infty)$
lok. min. $[e, e]$

$f'' = \frac{\frac{1}{x} \cdot \ln^2 x - (\ln x - 1) \cdot 2 \ln x \cdot \frac{1}{x}}{\ln^4 x} = \frac{\frac{1}{x} \cdot \ln x \left[\ln x - 2\ln x + 2\right]}{\ln^4 x} =$

$= \frac{\frac{1}{x} \cdot \cancel{\ln} x (2 - \ln x)}{\ln^4 x} = \frac{2 - \ln x}{x \cdot \ln^3 x}$

$2 - \ln x = 0$
$\ln x = 2$
$x = e^2$

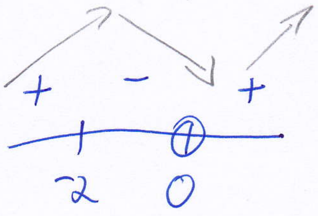


konkávní $(0,1), \langle e^2, \infty)$
konvexní $(1; e^2\rangle$
infl. bod $[e^2; \frac{e^2}{2}]$

- $f(x) = x + \ln x^2$ $D f = R \setminus \{0\}$

$f' = 1 + \frac{1}{x^2} \cdot 2x = 1 + \frac{2}{x}$

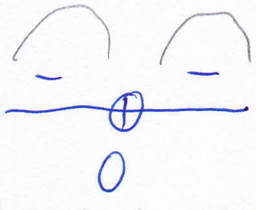
$1 + \frac{2}{x} = 0$
$\frac{2}{x} = -1$
$-2 = x$



kles. $\langle -2, 0)$
rost. $(-\infty; -2\rangle, (0; +\infty)$
lok. max $[-2; -2 + \ln 4]$

$f'' = -2x^{-2} = -\frac{2}{x^2}$

$-\frac{2}{x^2} = 0$
$-2 \neq 0$



konkávní $(-\infty, 0), (0; +\infty)$

- $f(x) = x^2 e^{-x}$ $\quad D f = R$

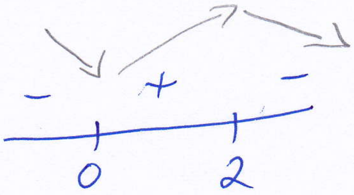
$f' = 2x e^{-x} + x^2 \cdot e^{-x}(-1) = x e^{-x}(2-x)$

$x e^{-x}(2-x) = 0$

$\downarrow \qquad \downarrow$

$x = 0 \qquad x = 2$

$e^{-x} > 0$



kles. $(-\infty; 0\rangle, \langle 2, \infty)$

rost. $\langle 0, 2\rangle$

lok. min. $[0; 0]$

lok. max. $[2; 4e^{-2}]$

$f'' = [x \cdot e^{-x}(2-x)]' = [e^{-x}(2x - x^2)]' = e^{-x}(-1)(2x - x^2)$

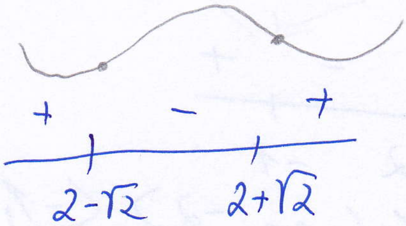
$+ e^{-x}(2-2x)\} = e^{-x}(-2x + x^2 + 2 - 2x) =$

$= e^{-x}(x^2 - 4x + 2)$

$x^2 - 4x + 2 = 0$

$D = 16 - 4 \cdot 2 = 8$

$x_{1,2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$



konvexní $(-\infty; 2-\sqrt{2}\rangle, \langle 2+\sqrt{2}; \infty)$

konkávní $\langle 2-\sqrt{2}, 2+\sqrt{2}\rangle$

~~lok. min.~~ $[2-\sqrt{2}; (6-4\sqrt{2}) \cdot e^{-2+\sqrt{2}}]$ } infl. body

~~lok. max.~~ $[2+\sqrt{2}; (6+4\sqrt{2}) \cdot e^{-2-\sqrt{2}}]$

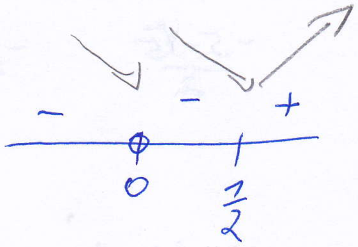
- $f(x) = x^2 \cdot e^{\frac{1}{x}}$ $\quad D f = R \setminus \{0\}$

$f' = 2x \cdot e^{\frac{1}{x}} + x^2 \cdot e^{\frac{1}{x}}(-x^{-2}) = x e^{\frac{1}{x}}\left(2 - x \cdot \frac{1}{x^2}\right) =$

$= x e^{\frac{1}{x}}\left(2 - \frac{1}{x}\right) = e^{\frac{1}{x}}(2x - 1)$

$2x - 1 = 0$

$x = \frac{1}{2}$



kles. $(-\infty; 0), (0; \frac{1}{2}\rangle$

rost. $\langle \frac{1}{2}, \infty)$

lok. min. $[\frac{1}{2}; \frac{1}{4}e^2]$

$f'' = e^{\frac{1}{x}} \cdot (-x^{-2})(2x-1) + e^{\frac{1}{x}} \cdot 2 = e^{\frac{1}{x}}\left(-\frac{1}{x^2}(2x-1)+2\right) =$

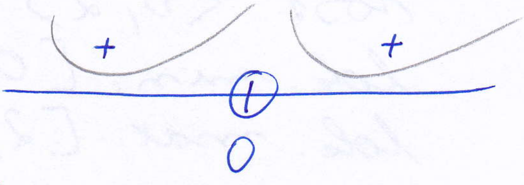
$= e^{\frac{1}{x}}\left(-\frac{2}{x} + \frac{1}{x^2} + 2\right)$

$-\frac{2}{x} + \frac{1}{x^2} + 2 = 0 \quad / x^2$

$-2x + 1 + 2x^2 = 0$

$2x^2 - 2x + 1 = 0$

$D = 4 - 4 \cdot 2 \cdot 1 = -4 \rightarrow$ nemá kořeny



~~konk~~. konkávní není

konvexní $(-\infty; 0), (0, \infty)$

- $f(x) = (x^2+x+1) \cdot e^x \quad Df = R$

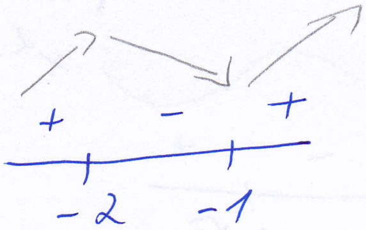
$f' = (2x+1)e^x + (x^2+x+1)e^x = e^x(2x+1+x^2+x+1) =$

$= e^x(x^2+3x+2)$

$x^2+3x+2 = 0$

$(x+2)(x+1) = 0$

$x = -2 \quad x = -1$



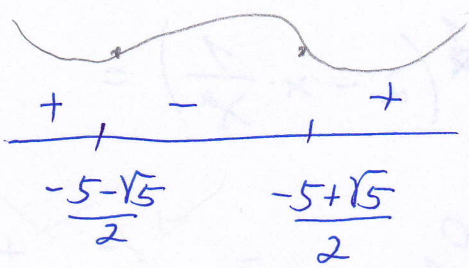
rost. $(-\infty; -2\rangle, \langle -1; +\infty)$

kles. $\langle -2; -1 \rangle$

lok. min. $[-1; e^{-1}]$

lok. max. $[-2; 3e^{-2}]$

$f'' = e^x(x^2+3x+2) + e^x(2x+3) = e^x(x^2+5x+5)$



$x^2+5x+5 = 0$

$D = 25 - 4 \cdot 5 = 5$

$x_{1,2} = \frac{-5 \pm \sqrt{5}}{2}$

konvexní $\left(-\infty; -\frac{5+\sqrt{5}}{2}\right\rangle, \left\langle \frac{-5+\sqrt{5}}{2}; +\infty\right)$

konkávní $\left\langle \frac{-5-\sqrt{5}}{2}; \frac{-5+\sqrt{5}}{2} \right\rangle$

infl. body $\left[\frac{-5-\sqrt{5}}{2}; (6+2\sqrt{5})e^{\frac{-5-\sqrt{5}}{2}}\right]; \left[\frac{-5+\sqrt{5}}{2}; (6-2\sqrt{5})e^{\frac{-5+\sqrt{5}}{2}}\right]$