UŽITÍ DERIVACE

1) Napište rovnici tečny dané funkce v daném bodě:

   a. $f(x) = \dfrac{x^2+4}{x-1}, T[2,?]$

   b. $f(x) = 3x^2 - 2\sqrt{x}, T[4,?]$

   c. $f(x) = \dfrac{x+3}{x-1}, T[2,?]$

2) Určete intervaly, na kterých je funkce rostoucí, klesající, konvexní, konkávní. Dále určete lokální extrémy a inflexní body.

   a. $f(x) = x^2(x-6)$

   b. $f(x) = x^4 - x^2$

   c. $f(x) = \dfrac{x}{(x+1)^2}$

   d. $f(x) = \dfrac{x^2}{x^2-4}$

   e. $f(x) = \dfrac{x}{\ln x}$

   f. $f(x) = x + \ln x^2$

   g. $f(x) = x^2 e^{-x}$

   h. $f(x) = x^2 e^{\frac{1}{x}}$

   i. $f(x) = (x^2+x+1)e^x$

užití derivace

Tečna:

- $f(x) = \frac{x^2+4}{x-1}$ $\quad T[2; ?]$

$y_0 = \frac{2^2+4}{2-1} = 8$

$f' = \frac{2x(x-1) - (x^2+4)}{(x-1)^2} = \frac{2x^2-2x-x^2-4}{(x-1)^2} = \frac{x^2-2x-4}{(x-1)^2}$

VZOREC:
$y = f'(x_0)\cdot(x-x_0)+y_0$

$f'(2) = \frac{4-4-4}{(2-1)^2} = -4$

$t: \; y = -4\cdot(x-2)+8$

$y = -4x+16$

- $f(x) = 3x^2 - 2\sqrt{x}$ $\quad T[4; ?]$

$y_0 = 3\cdot 4^2 - 2\sqrt{4} = 48-4 = 44$

$f' = 6x - 2\cdot\frac{1}{2}x^{-\frac{1}{2}} = 6x - \frac{1}{\sqrt{x}}$

$f'(4) = 6\cdot 4 - \frac{1}{\sqrt{4}} = 24 - \frac{1}{2} = \frac{47}{2}$

$t: \; y = \frac{47}{2}(x-4)+44$

$y = \frac{47}{2}x - 94 + 44$

$y = \frac{47}{2}x - 50$

- $f(x) = \frac{x+3}{x-1}$ $\quad T[2; ?]$

$y_0 = \frac{2+3}{2-1} = 5$

$f' = \frac{1(x-1)-(x+3)\cdot 1}{(x-1)^2} = \frac{-4}{(x-1)^2}$

$f'(x_0) = -\frac{4}{(2-1)^2} = -4$

$t: \; y = -4(x-2)+5$

$y = -4x+13$

monotonie + konvex. / konkáv.

- $f(x) = x^2 \cdot (x-6) = x^3 - 6x^2$

$D_f = \mathbb{R}$

$f' = 3x^2 - 12x$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$x = 0 \quad x = 4$

$+ \quad - \quad +$

$0 \quad 4$

rost. $(-\infty; 0\rangle, \langle 4, +\infty)$

kles. $\langle 0, 4\rangle$

lok. min. $[4; -32]$

lok. max. $[0; 0]$

$f'' = 6x - 12$

$6x - 12 = 0$

$6x = 12$

$x = 2$

$- \quad +$

2

konkávní $(-\infty; 2\rangle$

konvexní $\langle 2, +\infty)$

inflexní bod $[2; -16]$

- $f(x) = x^4 - x^2$

$D_f = \mathbb{R}$

$f' = 4x^3 - 2x$

$4x^3 - 2x = 0$

$x(4x^2 - 2) = 0$

$x = 0 \quad 4x^2 = 2$

$x^2 = \frac{1}{2}$

$x = \pm \frac{1}{\sqrt{2}}$

$- \quad + \quad - \quad +$

$-\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}$

rost. $\langle -\frac{1}{\sqrt{2}}, 0\rangle, \langle \frac{1}{\sqrt{2}}, +\infty)$

kles. $(-\infty, -\frac{1}{\sqrt{2}}\rangle, \langle 0, \frac{1}{\sqrt{2}}\rangle$

lok. min. $[-\frac{1}{\sqrt{2}}; -\frac{1}{4}], [\frac{1}{\sqrt{2}}; -\frac{1}{4}]$

lok. max. $[0; 0]$

$f'' = 13x^2 - 2$

$13x^2 = 2$

$x^2 = \frac{2}{13}$

$x = \pm\sqrt{\frac{2}{13}}$

$+ \quad - \quad +$

$-\sqrt{\frac{2}{13}} \quad \sqrt{\frac{2}{13}}$

konvexní $(-\infty; -\sqrt{\frac{2}{13}}\rangle, \langle \sqrt{\frac{2}{13}}, +\infty)$

konkávní $\langle -\sqrt{\frac{2}{13}}, \sqrt{\frac{2}{13}}\rangle$

infl. body: $[-\sqrt{\frac{2}{13}}; -\frac{22}{169}]$, $[\sqrt{\frac{2}{13}}; -\frac{22}{169}]$

- $f(x) = \frac{x}{(x+1)^2} \quad D_f = \mathbb{R} \setminus \{-1\}$

$f' = \frac{(x+1)^2 - x\,2(x+1)}{(x+1)^4} = \frac{x^2+2x+1-2x^2-2x}{(x+1)^4} = \frac{1-x^2}{(x+1)^4}$

$\frac{1-x^2}{(x+1)^4} = 0$

$1 - x^2 = 0$

$x = \pm 1$

$- \quad + \quad -$

$-1 \quad 1$

rost. $(-1; 1\rangle$

kles. $(-\infty; -1), \langle 1, +\infty)$

lok. max $[1; \frac{1}{4}]$

$$f'' = \frac{-2x(x+1)^4 - (1-x^2)(x+1)^3 \cdot 4}{(x+1)^8} = \frac{\cancel{(x+1)^3}\left[-2x(x+1) - 4(1-x^2)\right]}{(x+1)^{\cancel{8}}} =$$

$$= \frac{-2x^2 - 2x - 4 + 4x^2}{(x+1)^5} = \frac{2x^2 - 2x - 4}{(x+1)^5} = \frac{2(x^2 - x - 2)}{(x+1)^5} =$$

$$= \frac{2(x-2)\cancel{(x+1)}}{(x+1)^5} = \frac{2(x-2)}{(x+1)^4}$$

$2(x-2) = 0$

$x = 2$

$-$ $-$ $+$

-1 2

konvexní $\langle 2; +\infty)$

konkávní $(-\infty; -1)$, $(-1, 2\rangle$

infl. bod $[2; \frac{2}{9}]$

- $f(x) = \frac{x^2}{x^2-4}$ $\quad D f = R \setminus \{-2; 2\}$

$$f' = \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2} = \frac{2x^3 - 8x - 2x^3}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

$-8x = 0$

$x = 0$

$+$ $+$ $-$ $-$

-2 0 2

rost. $(-\infty; -2)$

$(-2; 0\rangle$

kles. $\langle 0; 2)$

$(2; +\infty)$

lok. max. $[0; 0]$

$$f'' = \frac{-8(x^2-4)^2 + 8x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} = \frac{\cancel{(x^2-4)}\left[-8x^2 + 32 + 32x^2\right]}{(x^2-4)^{\cancel{4}}} =$$

$$= \frac{24x^2 + 32}{(x^2-4)^3} = \frac{8(3x^2+4)}{(x^2-4)^3}$$

$8 \cdot (3x^2 + 4) = 0$

$3x^2 = -4$

$x^2 \neq -\frac{4}{3}$

nelze aby x^2 bylo záporné

$+$ $-$ $+$

-2 2

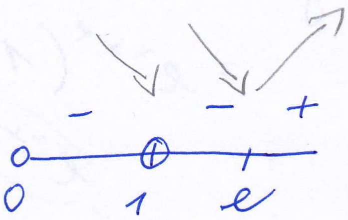
konkávní $(-2, 2)$

konvexní $(-\infty; -2)$, $(2; +\infty)$

- $f(x) = \dfrac{x}{\ln x}$ $\quad D_f = (0; +\infty) - \{1\}$

$\ln x \neq 0$
$x \neq e^0 \neq 1$

$f' = \dfrac{\ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \dfrac{\ln x - 1}{\ln^2 x}$ $\quad \ln x = 1$
$x = e$

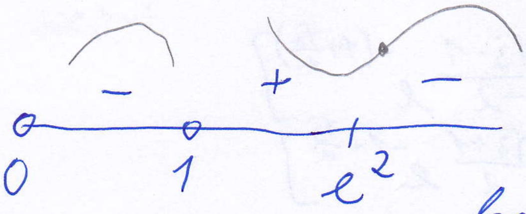


kles. $(0,1), (1,e\rangle$
rost. $\langle e, +\infty)$
lok. min. $[e, e]$

$f'' = \dfrac{\frac{1}{x} \cdot \ln^2 x - (\ln x - 1) \cdot 2\ln x \cdot \frac{1}{x}}{\ln^4 x} = \dfrac{\frac{1}{x} \cdot \ln x \left[\ln x - 2\ln x + 2\right]}{\ln^4 x} =$

$= \dfrac{\frac{1}{x} \cdot \cancel{\ln x}\,(2 - \ln x)}{\ln^4 x} = \dfrac{2 - \ln x}{x \cdot \ln^3 x}$

$2 - \ln x = 0$
$\ln x = 2$
$x = e^2$

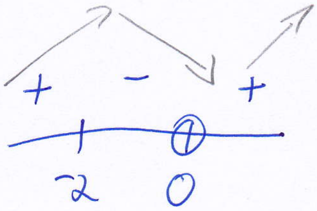


konkávní $(0,1), \langle e^2, \infty)$
konvexní $(1; e^2\rangle$
infl. bod $\left[e^2; \frac{e^2}{2}\right]$

- $f(x) = x + \ln x^2$ $\quad D_f = R \setminus \{0\}$

$f' = 1 + \dfrac{1}{x^2} \cdot 2x = 1 + \dfrac{2}{x}$

$1 + \dfrac{2}{x} = 0$
$\dfrac{2}{x} = -1$
$-2 = x$



kles. $\langle -2, 0)$
rost. $(-\infty; -2\rangle, (0; +\infty)$
lok. max $[-2; -2 + \ln 4]$

$f'' = -2x^{-2} = -\dfrac{2}{x^2}$ $\qquad -\dfrac{2}{x^2} = 0$
$-2 \neq 0$

konkávní $(-\infty, 0), (0; +\infty)$

- $f(x) = x^2 e^{-x}$ $\qquad D_f = \mathbb{R}$

$$f' = 2x e^{-x} + x^2 \cdot e^{-x}(-1) = x e^{-x}(2-x)$$

$$x e^{-x}(2-x) = 0 \qquad e^{-x} > 0$$

$x = 0 \qquad x = 2$

$-\quad 0 \quad +\quad 2 \quad -$

kles. $(-\infty; 0\rangle, \langle 2, \infty)$
rost. $\langle 0, 2\rangle$
lok. min. $[0; 0]$
lok. max. $[2; 4e^{-2}]$

$$f'' = [x \cdot e^{-x}(2-x)]' = [e^{-x}(2x - x^2)]' = e^{-x}(-1)(2x - x^2) + e^{-x}(2-2x) = e^{-x}(-2x + x^2 + 2 - 2x) =$$
$$= e^{-x}(x^2 - 4x + 2)$$

$$x^2 - 4x + 2 = 0$$
$$D = 16 - 4 \cdot 2 = 8$$
$$x_{1,2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$+\quad 2-\sqrt{2} \quad -\quad 2+\sqrt{2} \quad +$

konvexní $(-\infty; 2-\sqrt{2}\rangle, \langle 2+\sqrt{2}; \infty)$
konkávní $\langle 2-\sqrt{2}, 2+\sqrt{2}\rangle$
~~lok. min.~~ $[2-\sqrt{2}; (6-4\sqrt{2}) \cdot e^{-2+\sqrt{2}}]$
~~lok. max.~~ $[2+\sqrt{2}; (6+4\sqrt{2}) \cdot e^{-2-\sqrt{2}}]$
} infl. body

- $f(x) = x^2 \cdot e^{\frac{1}{x}}$ $\qquad D_f = \mathbb{R} \setminus \{0\}$

$$f' = 2x \cdot e^{\frac{1}{x}} + x^2 \cdot e^{\frac{1}{x}}(-x^{-2}) = x e^{\frac{1}{x}}\left(2 - x \cdot \frac{1}{x^2}\right) =$$
$$= x e^{\frac{1}{x}}\left(2 - \frac{1}{x}\right) = e^{\frac{1}{x}}(2x - 1)$$

$$2x - 1 = 0$$
$$x = \frac{1}{2}$$

$-\quad 0 \quad -\quad \frac{1}{2} \quad +$

kles. $(-\infty; 0), (0; \frac{1}{2}\rangle$
rost. $\langle \frac{1}{2}, \infty)$
lok. min. $[\frac{1}{2}; \frac{1}{4}e^2]$

$$f'' = e^{\frac{1}{x}} \cdot (-x^{-2})(2x-1) + e^{\frac{1}{x}} \cdot 2 = e^{\frac{1}{x}}\left(-\frac{1}{x^2}(2x-1)+2\right) =$$

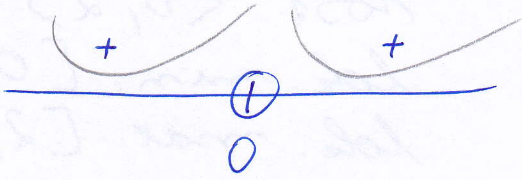
$$= e^{\frac{1}{x}}\left(-\frac{2}{x} + \frac{1}{x^2} + 2\right)$$

$$-\frac{2}{x} + \frac{1}{x^2} + 2 = 0 \quad / \cdot x^2$$

$$-2x + 1 + 2x^2 = 0$$

$$2x^2 - 2x + 1 = 0$$

$D = 4 - 4 \cdot 2 \cdot 1 = -4 \rightarrow$ nemá kořeny



~~kon~~. konkávní není

konvexní $(-\infty; 0), (0, \infty)$

• $f(x) = (x^2+x+1) \cdot e^x \quad Df = R$

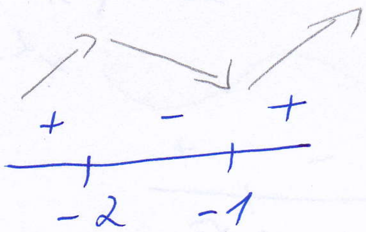
$$f' = (2x+1)e^x + (x^2+x+1)e^x = e^x(2x+1+x^2+x+1) =$$

$$= e^x(x^2+3x+2)$$

$$x^2+3x+2 = 0$$

$$(x+2)(x+1) = 0$$

$$x = -2 \quad x = -1$$



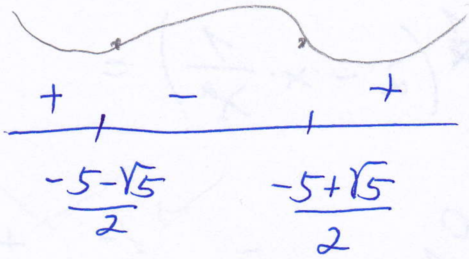
rost. $(-\infty; -2\rangle, \langle -1; +\infty)$

kles. $\langle -2; -1 \rangle$

lok. min. $[-1; e^{-1}]$

lok. max. $[-2; 3e^{-2}]$

$$f'' = e^x(x^2+3x+2) + e^x(2x+3) = e^x(x^2+5x+5)$$



$$x^2+5x+5 = 0$$

$$D = 25 - 4 \cdot 5 = 5$$

$$x_{1,2} = \frac{-5 \pm \sqrt{5}}{2}$$

konvexní $\left(-\infty; -\frac{5+\sqrt{5}}{2}\right\rangle, \left\langle \frac{-5+\sqrt{5}}{2}; +\infty\right)$

konkávní $\left\langle \frac{-5-\sqrt{5}}{2}; \frac{-5+\sqrt{5}}{2} \right\rangle$

infl. body $\left[\frac{-5-\sqrt{5}}{2}; (6+2\sqrt{5})e^{\frac{-5-\sqrt{5}}{2}}\right]; \left[\frac{-5+\sqrt{5}}{2}; (6-2\sqrt{5})e^{\frac{-5+\sqrt{5}}{2}}\right]$