UŽITÍ INTEGRÁLNÍHO POČTU

1) Vypočtěte objem rotačního tělesa, které vznikne rotací oblouku sinusoidy $y = \sin x$ kolem osy x na intervalu $\langle 0; \pi \rangle$.

$$\left[V = \frac{\pi^2}{2} \right]$$

2) Vypočtěte objem rotačního tělesa, které vznikne rotací útvaru ohraničeného křivkami: $y = 1 - x^2$, $y = x^2$ kolem osy x.

$$\left[V = \frac{2\sqrt{2}}{3}\pi \right]$$

3) Určete délku křivky $y = \dfrac{2\sqrt{x^3}}{3}$ na intervalu $\langle 0;1 \rangle$.

$$\left[L = \frac{4\sqrt{2}-2}{3} \right]$$

4) Určete obsah pláště rotačního kužele, který je tvořen úsečkou $y = 1 - x$ pro $x \in \langle 0;1 \rangle$.

$$\left[S = \pi\sqrt{2} \right]$$

5) Určete délku křivky $y = \sqrt{4 - x^2}$ na intervalu $\langle 0; \sqrt{2} \rangle$.

$$\left[L = \frac{\pi}{2} \right]$$

1) $V = \pi \cdot \int_0^{\pi} \sin^2 x \, dx$

$$\int \sin^2 x = \int \sin x \cdot \sin x \, dx = \left| \begin{matrix} u = \sin x & u' = \cos x \\ v' = \sin x & v = -\cos x \end{matrix} \right|$$

$$= -\sin x \cos x - \int -\cos^2 x \, dx =$$

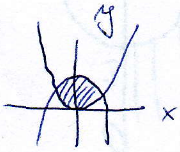
$$= -\sin x \cdot \cos x + \int \cos^2 x \, dx = -\sin x \cdot \cos x + \int 1 - \sin^2 x \, dx$$

$$\int \sin^2 x \, dx = -\sin x \cdot \cos x + x - \int \sin^2 x \, dx$$

$$2 \cdot \int \sin^2 x \, dx = -\sin x \cos x + x$$

$$\int \sin^2 x \, dx = \frac{-\sin x \cos x + x}{2}$$

$$V = \pi \cdot \left[\frac{-\sin x \cos x + x}{2} \right]_0^{\pi} = \pi \cdot \left(\frac{\pi}{2} - 0 \right) = \underline{\underline{\frac{\pi^2}{2}}}$$

2) 

$$1 - x^2 = x^2$$
$$1 = 2x^2$$
$$\frac{1}{2} = x^2$$
$$\pm \frac{1}{\sqrt{2}} = x$$

$$V_1 = \pi \cdot \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1 - x^2)^2 \, dx =$$

$$= \pi \cdot \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 1 - 2x^2 + x^4 \, dx =$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1/\sqrt{2}}^{1/\sqrt{2}} =$$

$$= \pi \left(\frac{1}{\sqrt{2}} - \frac{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^3}{3} + \frac{\left(\frac{1}{\sqrt{2}}\right)^5}{5} + \frac{1}{\sqrt{2}} + \frac{2\left(-\frac{1}{\sqrt{2}}\right)^3}{3} - \frac{\left(-\frac{1}{\sqrt{2}}\right)^5}{5} \right) =$$

$$= \pi \left(\frac{1}{\sqrt{2}} - \frac{2}{3} \cdot \frac{1}{2\sqrt{2}} + \frac{1}{5} \cdot \frac{1}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{2}{3} \cdot \frac{1}{2\sqrt{2}} + \frac{1}{5} \cdot \frac{1}{4\sqrt{2}} \right) =$$

$$= \pi \left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} \right) = \pi \cdot \frac{60 - 20 + 3 + 60 - 20 + 3}{60\sqrt{2}} = \pi \cdot \frac{86}{60\sqrt{2}}$$

$$V_2 = \pi \cdot \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (x^2)^2 \, dx = \pi \left[\frac{x^5}{5} \right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = \pi \cdot \left(\frac{1}{4\sqrt{2}} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{4\sqrt{2}} \right) = \frac{\pi}{10\sqrt{2}}$$

$$V = V_1 - V_2 = \frac{86\pi}{60\sqrt{2}} - \frac{\pi}{10\sqrt{2}} = \frac{43\pi}{30\sqrt{2}} - \frac{3\pi}{30\sqrt{2}} = \frac{40\pi}{30\sqrt{2}} =$$

$$= \frac{4\pi}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\pi\sqrt{2}}{3 \cdot 2} = \underline{\underline{\frac{2\pi\sqrt{2}}{3}}}$$

3)
$$L = \int \sqrt{1+(f')^2}\, dx$$

$$y = \frac{2}{3}\cdot x^{\frac{3}{2}} \rightarrow y' = \frac{2}{3}\cdot\frac{3}{2}\cdot x^{\frac{1}{2}} = \sqrt{x}$$

$$L = \int_0^1 \sqrt{1+(\sqrt{x})^2}\, dx = \int_0^1 \sqrt{1+x}\, dx = \int_0^1 (1+x)^{\frac{1}{2}}\, dx =$$

$$= \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \left[\frac{2}{3}\sqrt{(1+x)^3}\right]_0^1 = \frac{2}{3}\sqrt{2^3} - \frac{2}{3}\cdot 1 =$$

$$= \frac{2}{3}\cdot 2\sqrt{2} - \frac{2}{3} = \underline{\underline{\frac{4\sqrt{2}-2}{3}}}$$

4)
$$S = 2\pi \int f\cdot\sqrt{1+(f')^2}\, dx$$

$$y = 1-x \qquad y' = -1$$

$$S = 2\pi\cdot\int_0^1 (1-x)\cdot\sqrt{1+(-1)^2}\, dx = 2\pi\int_0^1 \sqrt{2}(1-x)\, dx =$$

$$= 2\sqrt{2}\pi\cdot\left[x - \frac{x^2}{2}\right]_0^1 = 2\sqrt{2}\pi\left(1 - \frac{1}{2} - 0\right) = \underline{\underline{\sqrt{2}\pi}}$$

5)
$$L = \int \sqrt{1+(f')^2}\, dx \qquad y = \sqrt{4-x^2}$$

$$y' = \frac{1}{2\sqrt{4-x^2}}\cdot(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

$$L = \int_0^{\sqrt{2}} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx = \int_0^{\sqrt{2}} \sqrt{1+\frac{x^2}{4-x^2}}\, dx = \int_0^{\sqrt{2}} \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx =$$

$$= \int_0^{\sqrt{2}} \frac{2}{\sqrt{4-x^2}}\, dx = \int_0^{\sqrt{2}} \frac{2}{2\sqrt{1-(\frac{x}{2})^2}}\, dx = \left[\frac{\arcsin\frac{x}{2}}{\frac{1}{2}}\right]_0^{\sqrt{2}} = \left[2\arcsin\frac{\sqrt{2}}{2} - 2\arcsin 0\right] =$$

$$= 2\cdot\frac{\pi}{4} - 0 = \underline{\underline{\frac{\pi}{2}}}$$