11. DIFERENCIÁLNÍ ROVNICE 1. ŘÁDU: SEPAROVATELNÉ ROVNICE, HOMOGENNÍ ROVNICE

CVIČENÍ Z MATEMATIKY 2/CF (DOPORUČENÉ ÚLOHY)

Standardní úlohy. Nalezněte obecné řešení separovatelné diferenciální rovnice.
(Jde o rovnici $y' = f(x)\,g(y)$.)

$y' = \frac{dy}{dx}$

1) $xy' - y = 0$	2) $xy' + y = 0$
3) $yy' + x = 0$	4) $y' = y$
5) $(1+x)y' = x(1-y)$	6) $y - xy' = 3(1 + x^2 y')$
7) $(x^2 + x)y' = 2y + 1$	8) $y' = e^{x+2y}$?

Nalezněte obecné řešení homogenní diferenciální rovnice.
(Jedná se o diferenciální rovnici typu $y' = F(\frac{y}{x})$. Substitucí $\frac{y}{x} = z$ ji převedeme na diferenciální rovnici s neznámou funkcí z proměnné x, která je řešitelná metodou separace proměnných.)

9) $x^2 y' = y^2 + xy$	10) $x^2 y' = x^2 + xy + y^2$
11) $y' = \dfrac{2x}{y} + \dfrac{y}{x}$	12) $xy' = y + y \ln \frac{y}{x}$

Výsledky.

1) $y = Cx$	2) $y = \dfrac{C}{x}$
3) $y = \pm\sqrt{C - x^2}, \quad C > 0$	4) $y = Ce^x$
5) $y = 1 - C(1+x)e^{-x}$	6) $y = 3 + \dfrac{Cx}{1+3x}$
7) $y = \dfrac{Cx^2}{(1+x)^2} - \frac{1}{2}$	8) $\cancel{y = -\ln\sqrt{\frac{C-e^x}{2}}}$ $y = -\frac{1}{2}\ln(C-2e^x) = \ln\frac{1}{\sqrt{C-2e^x}}$
9) $y = \dfrac{-x}{\ln\lvert x\rvert + C}$	10) $y = x\,\mathrm{tg}\,(\ln\lvert Cx\rvert)$
11) $y = \pm\sqrt{x^2 \ln x^4 + Cx^2}$	12) $y = x\,e^{Cx}$

DIF. ROVNICE 1. ŘÁDU, SEPARACE PROM., HOMOGENNÍ RCE

1) $xy' - y = 0$

$$x\frac{dy}{dx} - y = 0$$

$$x\frac{dy}{dx} = y \quad / dx$$

$$x\,dy = y\,dx$$

$$\int\frac{dy}{y} = \int\frac{dx}{x} \quad \text{| zintegrujeme}$$

$$\ln|y| = \ln|x| + \overset{\ln k}{c}$$

$$\ln|y| = \ln k|x|$$

$$|y| = k|x|$$

$$\underline{\underline{y = Cx}}$$

2) $xy' + y = 0$

$$\frac{x\,dy}{dx} = -y$$

$$\int\frac{dy}{y} = \int -\frac{dx}{x} \quad \text{| zintegrujeme}$$

$$\ln|y| = -\ln|x| + c$$

$$\ln|y| = \ln\frac{k}{|x|}$$

$$\underline{\underline{y = \frac{C}{x}}}$$

3) $yy' + x = 0$

$$y\cdot\frac{dy}{dx} = -x$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = C - x^2$$

$$\boxed{y_1 = \sqrt{C-x^2}}$$

$$\boxed{y_2 = -\sqrt{C-x^2}}$$

4) $y' = y$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx$$

$$\ln|y| = x + c$$

$$|y| = e^{x+c} = e^x e^c$$

$$\underline{\underline{y = Ce^x}}$$

5) $(1+x)y' = x(1-y)$

$$(1+x)\frac{dy}{dx} = x(1-y)$$

$$\int\frac{dy}{1-y} = \int\frac{x+1-1}{x+1}\,dx$$

$$-\ln|1-y| = x - \ln|x+1| + c$$

$$\ln|1-y| = \ln|x+1| - x + k$$

$$|1-y| = |x+1|\cdot e^{-x}\cdot e^{k}$$

$$1-y = C(x+1)e^{-x}$$

$$\underline{\underline{y = 1 - C(x+1)e^{-x}}}$$

6) $y - xy' = 3(1+x^2y')$

$$y - xy' = 3 + 3x^2y'$$

$$y - 3 = y'(3x^2+x)$$

$$y - 3 = \frac{dy}{dx}\cdot(3x^2+x)$$

$$\int\frac{dy}{y-3} = \int\frac{dx}{x(3x+1)}$$

$$\frac{1}{x(3x+1)} = \frac{A}{x} + \frac{B}{3x+1}$$

$$1 = A(3x+1) + Bx$$

$x=0$: $1 = A$

$x=-\frac{1}{3}$: $1 = -\frac{1}{3}B$

$B = -3$

$$\int\frac{dy}{y-3} = \int\left(\frac{1}{x} - \frac{3}{3x+1}\right)dx$$

$$\ln|y-3| = \ln|x| - \frac{\cancel{3}\ln|3x+1|}{\cancel{3}} + c$$

$$|y-3| = \frac{|x|}{|3x+1|}\cdot e^c$$

$$y-3 = C\cdot\frac{x}{3x+1}$$

$$\underline{\underline{y = C\cdot\frac{x}{3x+1} + 3}}$$

7) $(x^2+x)y' = 2y+1$

$(x^2+x)\frac{dy}{dx} = 2y+1$

$\int\frac{dy}{2y+1} = \int\frac{dx}{x^2+x}$

$\frac{\ln|2y+1|}{2} = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$

$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$

$1 = A(x+1) + Bx$

$x=0: \quad 1 = A$

$x=-1: \quad 1 = -B$

$B = -1$

$\ln|2x+1| = 2\ln|x| - 2\ln|x+1| + c$

$|2y+1| = \frac{x^2}{(x+1)^2}\cdot e^c$

$2y+1 = k\cdot\frac{x^2}{(x+1)^2}$

$2y = k\cdot\frac{x^2}{(x+1)^2} - 1$

$y = C\cdot\frac{x^2}{(x+1)^2} - \frac{1}{2}$

8) $y' = e^{x+2y}$

$\frac{dy}{dx} = e^x\cdot e^{2y}$

$\int e^{-2y}dy = \int e^x dx$

$\frac{e^{-2y}}{-2} = e^x + k$

$e^{-2y} = -2e^x + K \quad / \ln$

$-2y = \ln(C - 2e^x)$

$y = -\frac{1}{2}\ln(C-2e^x) = \ln\frac{1}{\sqrt{C-2e^x}}$

9) $x^2y' = y^2 + xy \quad / : x^2$

$y' = \frac{y^2}{x^2} + \frac{xy}{x^2}$

$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$

$xz' + \cancel{z} = z^2 + \cancel{z}$

$x\frac{dz}{dx} = z^2$

$z^{-2}dz = \frac{dx}{x}$

$-z^{-1} = \ln|x| + k$

$-\frac{x}{y} = \ln|x| + k$

$y = -\frac{x}{\ln|x| + C}$

Použijeme substituci

$\frac{y}{x} = z$

$\frac{y'\cdot x - y}{x^2} = z'$

$y'x - y = xz'$

$y' = xz' + \frac{y}{x}$

$y' = xz' + z$

10) $x^2y' = x^2 + xy + y^2 \quad / : x^2$

$y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$

subs.: $\frac{y}{x} = z$

$y' = xz' + z$

$xz' + \cancel{z} = 1 + \cancel{z} + z^2$

$\frac{dz}{1+z^2} = \frac{dx}{x}$

$\operatorname{arctg} z = \ln|x| + k$

$\frac{y}{x} = \operatorname{tg}(\ln|x| + \ln C)$

$y = x\cdot\operatorname{tg}(\ln|C\cdot x|)$

11) $y' = \frac{2x}{y} + \frac{y}{x}$ | subs: $\frac{y}{x} = z$, $y' = xz' + z$

$x z' + \cancel{z} = \frac{2}{z} + \cancel{z}$

$\int \frac{1}{2} z \, dz = \int \frac{dx}{x}$

$\frac{1}{2} \frac{z^2}{2} = \ln|x| + k$

$\frac{y^2}{4x^2} = \ln|x| + k$

$y^2 = 4x^2 \ln|x| + 4kx^2$

$y = \pm\sqrt{x^2 \ln x^4 + Cx^2}$

12) $xy' = y + y \ln \frac{y}{x}$ | $:x$

$y' = \frac{y}{x} + \frac{y}{x} \ln \frac{y}{x}$ | $z = \frac{y}{x}$, $y' = xz' + z$

$x z' + \cancel{z} = z + z \ln z$

$x \frac{dz}{dx} = z \ln z$

$\int \frac{dz}{z \ln z} = \int \frac{dx}{x}$

| $\ln z = t$
| $\frac{1}{z} dz = dt$
| $\int \frac{1}{\ln z} \cdot \frac{1}{z} dz = \int \frac{1}{t} dt = \ln|t|$
| $= \ln|\ln z|$

$\ln|\ln z| = \ln|x| + k$

$|\ln z| = |x| \cdot e^{k}$

$\ln z = Kx$

$z = e^{kx}$

$\frac{y}{x} = e^{kx}$

$y = x \cdot e^{Cx}$