

OPAKOVÁNÍ:

I. Napište definiční obor:

a) $f(x, y) = \ln \frac{y - x^2 - 1}{x^2 + y^2 - 9}$

b) $f(x, y) = \sqrt{\frac{x^2 + 2y^2 - 4}{y + x}}$

c) $f(x, y) = \ln \frac{x^2 - 1}{x^2 + y^2 - 9}$

d) $f(x, y) = \arccos(y+1) - \sqrt{16 - x^2 - y^2}$

II. Vypočítejte parciální derivace

a) $f(x, y) = x^2 y \cdot \ln(x - 2y) + \sin \frac{\pi}{4}$ $f'_x = ?$ $f'_y = ?$

b) $f(x, y) = x \cdot \arctg \sqrt{y-2} - \arctg 6$ $f''_{xy} = ?$

c) $f(x, y) = \sqrt{y} \cdot \sin(x^2 + 3y) - e^5 + \sqrt{3}$ $f'_y = ?$ $f''_{xx} = ?$

d) $f(x, y) = \frac{x}{\sqrt{x^2 - y^2}} - \frac{x + 3y}{e^x - 5}$ $f'_x = ?$ $f'_y = ?$

e) $f(x, y) = x^2 \cdot e^{x-2y} + \sin(\pi y)$ $f'_x(M) = ?$ $f'_y(M) = ?$
 $M[2, 1]$

III. Napište rovnici tečné roviny a normály ke grafu funkce z v bodě T

a) $z = \arctg \frac{2-3x}{3+2y}$ $T[1; -1; ?]$

b) $z = (x+y)e^{x^2-xy}$ $T[1; 1; ?]$

c) $\arctg y - yx^3 - x \ln z + 1 = 0$ $T[1; 0; e^2]$

d) $x^3 - 3x^2 z + y^2 z^3 - \sqrt{y} \cdot x^2 = 0$ $T[2; 4; 0]$

IV. Vypočítejte lokální extrémny funkce:

a) $f(x,y) = x^2 + 6y^2 - xy + 10x + 18y - 2$

b) $f(x,y) = 4 - 2x^2 - 3y^2 + 2xy + 12x + 4y$

c) $f(x,y) = 4^3 + 9xy^2 + 15x^2 + 27y^2$

d) $f(x,y) = x^2y^2 - x^2 - y^2$

V. Vypočítejte extrémny vázane podmínkou M:

a) $f(x,y) = 3x - 6y - 4$ M: $y - \ln \sqrt{x} - 3 = 0$

b) $f(x,y) = x^2 + 2y^2$ M: $x^2 - 2x + 2y^2 + 4y = 0$

c) $f(x,y) = e^x - y + 3$ M: $y + (1-x)e^2 = 0$

d) $f(x,y) = 6 - 4x - 3y$ M: $x^2 + y^2 - 1 = 0$

e) $f(x,y) = x^2 - y^2$ M: $2x - y + 1 = 0$

f) $f(x,y) = x^2 + 12xy + 2y^2$ M: $4x^2 + y^2 - 25 = 0$

g) $f(x,y) = 3y + 2x^4 + 9x^2 + 6$ M: $-x^4 - y + 3x^2 - 2 = 0$

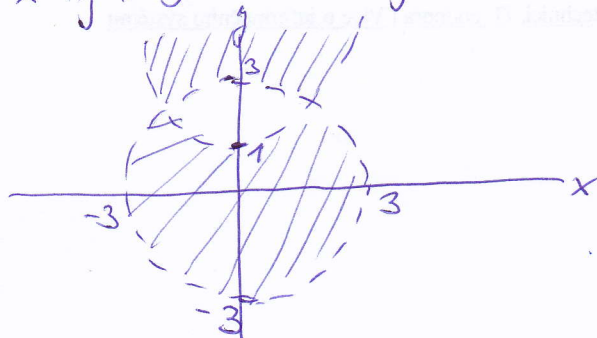
h) $f(x,y) = \ln(x^2 + y^2)$ M: $-y + x + 3 = 0$

i) $f(x,y) = \sqrt{4x + y^2 + 5}$ M: $y - 2x + 3 = 0$

Řešení:

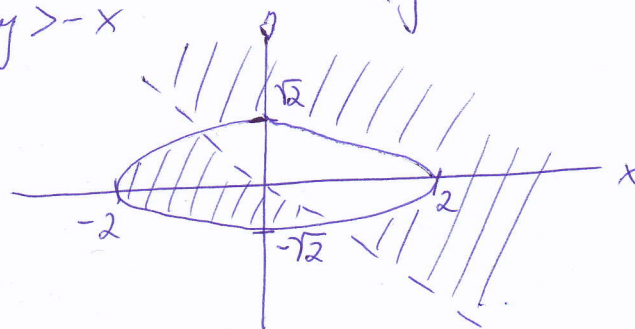
I. a) $\frac{y - x^2 - 1}{x^2 + y^2 - 9} > 0$

$$\begin{aligned} \downarrow \frac{+}{+} & \Rightarrow \downarrow \\ y & > x^2 + 1 \\ x^2 + y^2 & > 9 \end{aligned} \quad \begin{aligned} \downarrow \frac{-}{-} & \Rightarrow \downarrow \\ y & < x^2 + 1 \\ x^2 + y^2 & < 9 \end{aligned}$$



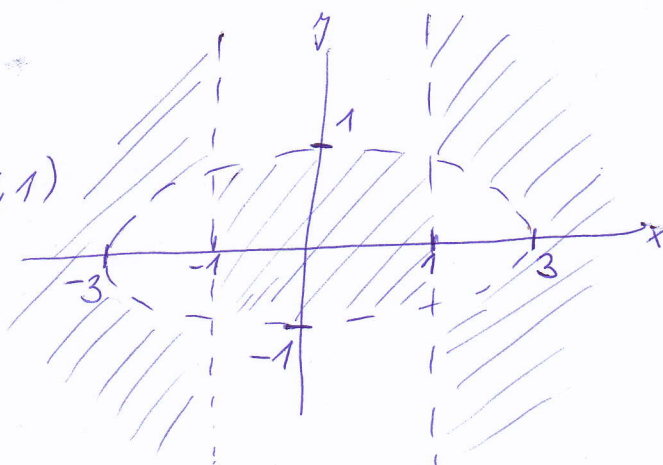
b) $\frac{x^2 + 2y^2 - 4}{y + x} \geq 0$

$$\begin{aligned} \downarrow \frac{+}{+} & \Rightarrow \downarrow \\ x^2 + 2y^2 & \geq 4 \\ \frac{x^2}{4} + \frac{y^2}{2} & \geq 1 \\ y & > -x \end{aligned} \quad \begin{aligned} \downarrow \frac{-}{-} & \Rightarrow \downarrow \\ \frac{x^2}{4} + \frac{y^2}{2} & \leq 1 \\ y & < -x \end{aligned}$$

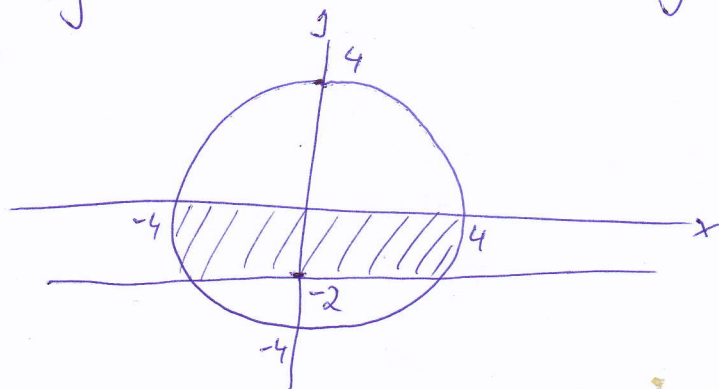


c) $\frac{x^2 - 1}{x^2 + 9y^2 - 9} > 0$

$$\begin{aligned} \downarrow \frac{+}{+} & \Rightarrow \downarrow \\ x^2 & > 1 \Rightarrow (-\infty, -1) \cup (1, \infty) \\ x^2 + 9y^2 & > 9 \\ \frac{x^2}{9} + \frac{y^2}{1} & > 1 \end{aligned} \quad \begin{aligned} \downarrow \frac{-}{-} & \Rightarrow \downarrow \\ x^2 & < 1 \Rightarrow x \in (-1, 1) \\ \frac{x^2}{9} + \frac{y^2}{1} & < 1 \end{aligned}$$



d) $-1 \leq y + 1 \leq 1$ $16 - x^2 - y^2 \geq 0$
 $-2 \leq y \leq 0$ $16 \geq x^2 + y^2$



II.

$$a) f'_x = 2x \cdot y \cdot \ln(x-2y) + x^2 \cdot y \cdot \frac{1}{x-2y}$$

$$f'_y = x^2 \ln(x-2y) + x^2 y \cdot \frac{1}{x-2y} \cdot (-2)$$

$$b) f'_x = \arctg \sqrt{y-2}$$

$$f''_{xy} = \frac{1}{1+(\sqrt{y-2})^2} \cdot \frac{1}{2\sqrt{y-2}} = \frac{1}{y-1} \cdot \frac{1}{2\sqrt{y-2}}$$

$$c) f'_y = \frac{1}{2\sqrt{y}} \sin(x^2+3y) + \sqrt{y} \cdot \cos(x^2+3y) \cdot 3$$

$$f'_x = \sqrt{y} \cos(x^2+3y) \cdot 2x$$

$$f''_{xx} = -\sqrt{y} \sin(x^2+3y) \cdot 2x \cdot 2x + \sqrt{y} \cos(x^2+3y) \cdot 2$$

$$d) f'_x = \frac{\sqrt{x^2-y^2} - x \cdot \frac{1}{2\sqrt{x^2-y^2}} \cdot 2x}{x^2-y^2} - \frac{e^x - 5 - (x+3y)e^x}{(e^x-5)^2} =$$

$$= \frac{\sqrt{x^2-y^2} - \frac{x^2}{\sqrt{x^2-y^2}}}{x^2-y^2} - \frac{e^x - 5 - x e^x - 3y e^x}{(e^x-5)^2} =$$

$$= \frac{\frac{x^2-y^2-x^2}{\sqrt{x^2-y^2}}}{x^2-y^2} - \frac{e^x(1-x-3y)-5}{(e^x-5)^2} =$$

$$= \frac{-y^2}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{e^x(1-x-3y)-5}{(e^x-5)^2}$$

$$f'_y = \frac{0 - x \cdot \frac{1}{2\sqrt{x^2-y^2}} (2y)}{x^2-y^2} - \frac{3}{e^x-5} = \frac{-xy}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{3}{e^x-5}$$

$$e) f'_x = 2x e^{x-2y} + x^2 e^{x-2y}$$

$$f'_x(1) = 2 \cdot 2 \cdot e^{2-2} - 4e^{2-2} = 4 \cdot 1 - 4 \cdot 1 = \underline{\underline{0}}$$

$$f'_y = x^2 e^{x-2y} \cdot (-2) + \cos(\pi y) \cdot \pi$$

$$f'_y(1) = 4 \cdot e^{2-2} \cdot (-2) + \cos \pi \cdot \pi = -8 + (-1)\pi = \underline{\underline{-8-\pi}}$$

III.

$$a) \quad R - R_0 = f'_x(T) \cdot (x - x_0) + f'_y(T) \cdot (y - y_0)$$

$$x_0 = 1 \quad y_0 = -1 \quad R_0 = \arctan \frac{2-3}{3-2} = \arctan(-1) = -\frac{\pi}{4}$$

$$f'_x = \frac{1}{1 + \left(\frac{2-3x}{3+2y}\right)^2} \cdot \frac{-3}{3+2y} \quad f'_x(T) = \frac{1}{1 + \left(\frac{2-3}{3-2}\right)^2} \cdot \frac{-3}{3-2} = \frac{1 \cdot (-3)}{2} = -\frac{3}{2}$$

$$f'_y = \frac{1}{1 + \left(\frac{2-3x}{3+2y}\right)^2} \cdot \frac{-(2-3x) \cdot 2}{(3+2y)^2} \quad f'_y(T) = \frac{1}{1 + \left(\frac{2-3}{3-2}\right)^2} \cdot \frac{-(2-3) \cdot 2}{(3-2)^2} = 1$$

$$R + \frac{\pi}{4} = -\frac{3}{2}(x-1) + 1(y+1)$$

$$\boxed{\lambda: R = -\frac{3}{2}x + y - \frac{\pi}{4} + \frac{5}{2}}$$

$$-\frac{3}{2}x + y - R - \frac{\pi}{4} + \frac{5}{2} = 0$$

$$\vec{n} \left(-\frac{3}{2}, 1, -1 \right)$$



$$\boxed{\begin{aligned} m: \quad x &= 1 - \frac{3}{2}\lambda \\ y &= -1 + \lambda \\ R &= -\frac{\pi}{4} - \lambda \end{aligned} \quad \lambda \in \mathbb{R}}$$

$$b) \quad x_0 = 1 \quad y_0 = 1 \quad R_0 = (1+1)e^{1-1} = 2$$

$$f'_x = e^{x^2-xy} + (x+y)e^{x^2-xy}(2x-y) \quad f'_x(T) = e^0 + 2e^0(2-1) = 3$$

$$f'_y = e^{x^2-xy} + (x+y)e^{x^2-xy}(-x) \quad f'_y(T) = e^0 + 2e^0(-1) = -1$$

$$R - 2 = 3(x-1) - 1(y-1)$$

$$\boxed{\lambda: R = 3x - y}$$

$$3x - y - R = 0$$

$$\vec{n} \left(3, -1, -1 \right)$$

$$\boxed{\begin{aligned} m: \quad x &= 1 + 3\lambda \\ y &= 1 - \lambda \\ R &= 2 - \lambda \end{aligned} \quad \lambda \in \mathbb{R}}$$

$$c) \quad F'_x = -y \cdot 3x^2 - \ln x \quad F'_x(T) = -\ln e^2 = -2$$

$$F'_y = \frac{1}{1+y^2} - x^3 \quad F'_y(T) = 1 - 1 = 0$$

$$F'_z = -x \cdot \frac{1}{z} \quad F'_z(T) = -1 \cdot \frac{1}{e^2} = -\frac{1}{e^2}$$

$$f'_x = -\frac{F'_x}{F'_z} = \frac{2}{-\frac{1}{e^2}} = -2e^2$$

$$f'_y = -\frac{F'_y}{F'_z} = -\frac{0}{-\frac{1}{e^2}} = 0$$

$$2 - e^2 = -2e^2(x-1) + 0(y-0)$$

$$L: \quad 2 = -2e^2 x + 3e^2$$

$$-2e^2 x + 3e^2 - 2 = 0$$

$$\vec{n} = (-2e^2, 0, -1)$$

$$M: \quad \begin{aligned} x &= 1 - 2e^2 \lambda \\ y &= 0 \\ z &= e^2 - 1 \quad \lambda \in \mathbb{R} \end{aligned}$$

$$d) \quad F'_x = 3x^2 - 6xz - 2\sqrt{y} \cdot x \quad F'_x(T) = 12 - 0 - 8 = 4$$

$$F'_y = 2yz^3 - \frac{x^2}{2\sqrt{y}} \quad F'_y(T) = 0 - \frac{4}{4} = -1$$

$$F'_z = -3x^2 + yz \cdot 3z^2 \quad F'_z(T) = -12 + 0 = -12$$

$$f'_x = -\frac{F'_x}{F'_z} = -\frac{4}{-12} = \frac{1}{3}$$

$$f'_y = -\frac{F'_y}{F'_z} = -\frac{-1}{-12} = -\frac{1}{12}$$

$$2 - 0 = \frac{1}{3}(x-2) - \frac{1}{12}(y-4)$$

$$L: \quad 2 = \frac{1}{3}x - \frac{1}{12}y - \frac{1}{3}$$

$$\frac{1}{3}x - \frac{1}{12}y - 2 - \frac{1}{3} = 0$$

$$\vec{n} = \left(\frac{1}{3}, -\frac{1}{12}, -1\right)$$

$$M: \quad \begin{aligned} x &= 2 + \frac{1}{3}\lambda \\ y &= 4 - \frac{1}{12}\lambda \\ z &= -1 \quad \lambda \in \mathbb{R} \end{aligned}$$

IV.

$$a) \begin{aligned} f'_x &= 2x - y + 10 \\ f'_y &= 12y - x + 18 \end{aligned}$$

$$2x - y + 10 = 0$$

$$12y - x + 18 = 0 \Rightarrow 12y + 18 = x$$

$$2 \cdot (12y + 18) - y + 10 = 0$$

$$24y + 36 - y + 10 = 0$$

$$23y = -46$$

$$y = -2$$

$$x = 12 \cdot (-2) + 18 = -6$$

$$f''_{xx} = 2$$

$$f''_{xy} = -1$$

$$f''_{yx} = -1$$

$$f''_{yy} = 12$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 12 \end{vmatrix}$$

$$D_2 = 2 \cdot 12 - 1 = 23 > 0 \quad [-6, -2]$$

$$f''_{xx} > 0$$

lok. min. ~~max~~

b) lok. max. $[4, 2]$

c) lok. min. $[0, 0]$, lok. max. $[-5, 0]$
(sedlo: $[-3, 2], [-3, -2]$)

d) lok. max. $[0, 0]$

(sedlo: $[1, 1], [-1, 1], [1, -1], [-1, -1]$)

V. a) ZM lze vyjádřit y , použijí dosazovací metodu

$$y - \ln \sqrt{x} - 3 = 0$$

$$y = \ln \sqrt{x} + 3$$

$$h(x) = 3x - 6(\ln \sqrt{x} + 3) - 4 = 3x - 6 \ln \sqrt{x} - 22$$

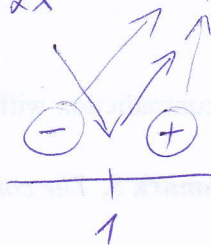
$$h'(x) = 3 - \frac{6}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = 3 - \frac{6}{2x} = 3 - \frac{3}{x}$$

$$3 - \frac{6}{2x} = 0 \quad | \cdot x$$

$$3x - 3 = 0$$

$$x = 1$$

$$y = \ln 1 + 3 = 3$$



nášane minimum

$[1, 3]$

b) z M nelze nic vyjádřit, proto použijí jacobian:

$$\begin{vmatrix} f'_x & f'_y \\ g'_x & g'_y \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2x & 4y \\ 2x-2 & 4y+4 \end{vmatrix} = 0$$

$$2x(4y+4) - 4y(2x-2) = 0$$

$$8xy + 8x - 8xy + 8y = 0$$

$$8x = -8y$$

$$x = -y$$

dosadíme do M:

$$x^2 - 2x + 2y^2 + 4y = 0$$

$$y^2 + 2y + 2y^2 + 4y = 0$$

$$3y^2 + 6y = 0$$

$$3y(y+2) = 0$$

$$y_1 = 0 \quad y_2 = -2$$

$$x_1 = 0 \quad x_2 = 2$$

$$[0, 0] \quad [2, -2]$$

$$f(0, 0) = 0^2 + 2 \cdot 0^2 = 0 \quad f(2, -2) = 4 + 2 \cdot 4 = 12$$

naš. MIN

naš. MAX

c) vyjádříme M:

$$y = -(1-x)e^2$$

$$h(x) = e^x + (1-x)e^2 + 3 = e^x + e^2 - xe^2 + 3$$

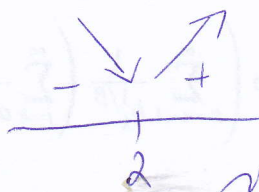
$$h'(x) = e^x - e^2$$

$$e^x - e^2 = 0$$

$$e^x = e^2$$

$$x = 2$$

$$y = -(1-2)e^2 = e^2$$



naš. min. $[2, e^2]$

d) val's. max. $[-\frac{4}{5}, -\frac{3}{5}]$, val's. min. $[\frac{4}{5}, \frac{3}{5}]$

e) val's. max. $[-\frac{2}{3}, -\frac{1}{3}]$

f) val's. min. $[-2, 3], [2, -3]$
val's. max. $[\frac{3}{2}, 4], [-\frac{3}{2}, -4]$

g) val's. min $[0, -2]$
val's. max $[3, -56], [-3, 56]$

h) val's. min. $[-\frac{3}{2}, \frac{3}{2}]$

i) val's. min. $[1, -1]$